Robust Subspace Segmentation by Low-Rank Representation (LRR)

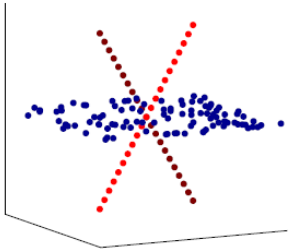
Written by Xu Qian

2016.11.24

Liu G, Lin Z, Yu Y. Robust Subspace Segmentation by Low-Rank Representation[C]// International Conference on Machine Learning. 2010:663-670.

Background

- *Subspace Segmentation*
  - Hypothesis
    - data as lying near several sub-spaces
    - low-rank
    - independent
  - Goal: segment data into clusters with each cluster corresponding to a subspace
  - Method: LRR



A mixture of subspaces
- a 2D plane
- two 1D lines

- *Low-Rank Representation (LRR)*
  - Given a set of data vectors, LRR seeks the lowest-rank representation among all the candidates that represent all vectors as the linear combination of the bases in a dictionary.

- *SR & LRR*
  - *Sparse Representation (SR):* compute the sparsest representation of each data vector individually
  - *low-Rank Representation (LRR):* find the lowest-rank representation of a collection of vectors jointly
    - captures the global structure of data
    - robust subspace segmentation from corrupted data

Low-Rank Representation (LRR)

Notations:

- data: $X = [x_1, x_2, \cdots, x_n] \quad X \in \mathbb{R}^{d \times n}$
- *k* subspaces: $\{\mathcal{S}_i\}_{i=1}^{k}$
- dimensions: $\{d_i\}_{i=1}^{k}$

Model:

$$\min_{Z} ||Z||_*, \quad \text{s.t.}, \ X = XZ.$$

the observed data matrix itself is chosen as the dictionary

$$\min_{Z,E} ||Z||_* + \lambda ||E||_{2,1}, \quad \text{s.t.}, \ X = XZ + E,$$

$||Z||_*$ low rank constraint
nuclear norm: the sum of the singular values of the matrix

$||E||_{2,1}$ columns sparsity; the corruptions are sample-specific
i.e. some data vectors are corrupted and the others are clean

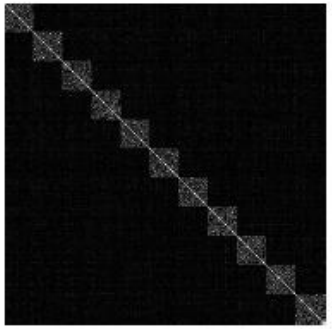
Low-Rank Representation (LRR)

optimal solution Z*

$$Z_i^* \ \textit{being an} \ n_i \times n_i \ \textit{matrix} \qquad rank(Z_i^*) = d_i, \ \forall i.$$

$$Z^* = \begin{bmatrix} Z_1^* & 0 & 0 & 0 \\ 0 & Z_2^* & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & Z_k^* \end{bmatrix}_{n \times n}$$

block-diagonal



recover the original data

After obtaining an optimal solution (Z^*, E^*)
recover the original data by using $X - E^*$ (or XZ^*)

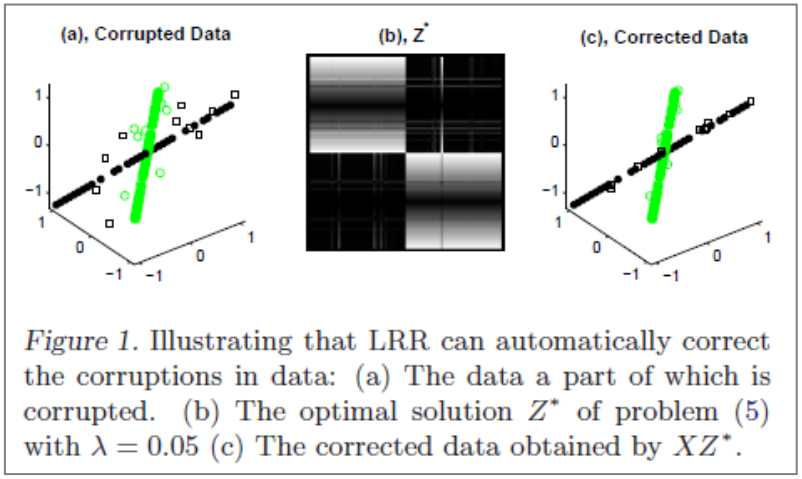
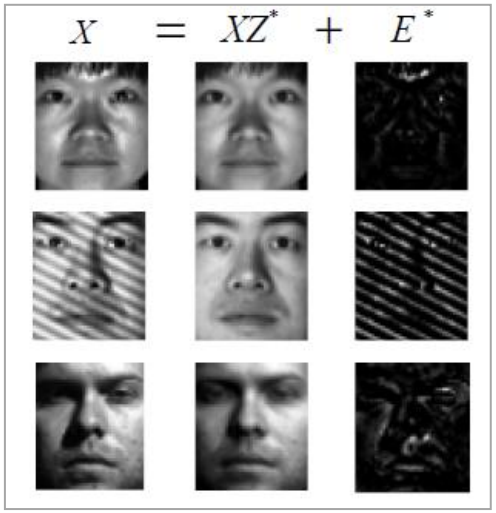


Figure 1. Illustrating that LRR can automatically correct the corruptions in data: (a) The data a part of which is corrupted. (b) The optimal solution Z^* of problem (5) with $\lambda = 0.05$ (c) The corrected data obtained by XZ^*.

original data | corrected data | error



Latent Low-Rank Representation for Subspace Segmentation and Feature Extraction (LatLRR)

Liu G, Yan S. Latent Low-Rank Representation for subspace segmentation and feature extraction[C]// International Conference on Computer Vision. IEEE Computer Society, 2011:1615-1622.

Latent Low-Rank Representation (LatLRR)

LRR

$$\min_{Z} \quad \|Z\|_*, \quad \text{s.t.} \quad X_O = X_O Z.$$

X_O denotes the observed data

the observed data matrix itself is chosen as the dictionary; requires the sampling of the dictionary being sufficient

However： observations are insufficient and/or corrupted

LatLRR

$$\min_{Z} \|Z\|_*, \quad \text{s.t.} \quad X_O = [X_O, X_H]Z,$$

X_H the unobserved, hidden data

X_O and X_H dictionary

use both observed and hidden data as dictionary

Hidden Effects Recovery: recover $Z^*_{O|H}$ in the absence of the hidden data X_H.

integrates subspace segmentation and feature extraction into a unified framework

Latent Low-Rank Representation (LatLRR)

Recovering Hidden Effects

an approximate recovery by analyzing the properties of the hidden effects

$$\text{Let } L^*_{H|O} = U\Sigma V_H^T V_H \Sigma^{-1} U^T \quad \Rightarrow \quad X_O = X_O Z^*_{O|H} + L^*_{H|O} X_O$$

Noiseless Data:

$$\min_{Z,L} \|Z\|_* + \|L\|_*, \text{ s.t. } X = XZ + LX.$$

Corrupted Data:

$$\begin{aligned} \min_{Z,L,E} \quad & \|Z\|_* + \|L\|_* + \lambda \|E\|_1 \\ \text{s.t.} \quad & X = XZ + LX + E, \end{aligned}$$



(a) $Z^*_{O|H}$ (b) LRR (c) LatLRR

the recovery of the hidden effects

LatLRR reconstructs data matrix X from two directions: column (XZ) and row (LX).

By combining both directions, LatLRR can defend missed points and be more robust against noise.

Latent Low-Rank Representation (LatLRR)

Subspace Segmentation

Algorithm 2 Subspace Segmentation by LRR

Input: data matrix X, number of subspaces k

1. obtain the lowest-rank representation by solving problem (5)
2. construct an undirected graph by using the lowest-rank representation to define the affinity matrix of the graph
3. use NCut to segment the vertices of the graph into k clusters

Feature Extraction

$$X = XZ^* + L^*X + E^*$$

data = principal features + salient features + sparse noise

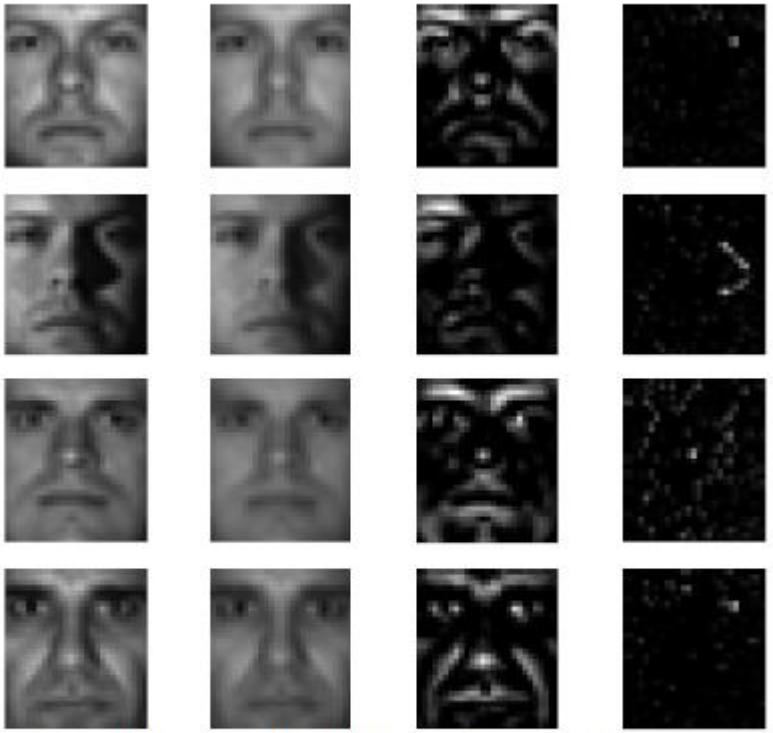


Figure 4. **Illustrating LatLRR's mechanism of decomposing the data.** Given a data matrix X, LatLRR decomposes it into a low-rank part XZ^* that represents the principal features, a low-rank part L^*X that encodes the salient features and a sparse part E^* that fits noise. These examples are selected from the training set.

Appendix - Multiple Representation Methods

$$X = AZ,$$

- data: $X = [x_1, x_2, \cdots, x_n] \quad X \in \mathbb{R}^{d \times n}$
- dictionary: $A = [a_1, a_2, \cdots, a_m] \quad A \in \mathbb{R}^{d \times m}$
- coefficient matrix $Z = [z_1, z_2, \cdots, z_n] \quad Z \in \mathbb{R}^{m \times n}$

each *zi* being the *representation* of *xi*

Dictionary Learning

$$X = AZ$$

Self-Representation

- sample-level: $X = XZ$, $X \in \mathbb{R}^{d \times n}, Z \in \mathbb{R}^{n \times n}$
- feature-level: $X = XZ$, $X \in \mathbb{R}^{n \times d}, Z \in \mathbb{R}^{d \times d}$

dictionary: X

Sparse Representation (SR)

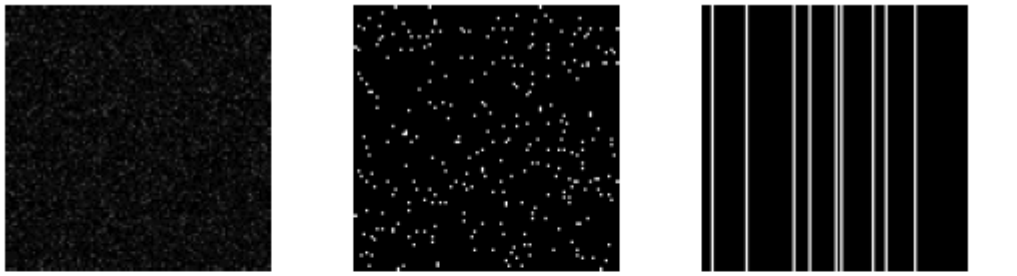
$$\|Z\|_{sparse}$$
$$s.t. \;\; X = AZ$$

Low-Rank Representation (LRR)

$$min \;\; rank(Z)$$
$$s.t. \;\; X = AZ$$

Appendix - About Error

Notice that the word “error” generally refers to the deviation between model assumption (i.e., subspaces) and data. It could exhibit as noise [6], missed entries [6], outliers [16] and corruptions [7] in reality.



(a) noise (b) random corruptions (c) sample–specific corruptions

Fig. 2. **Illustrating three typical types of errors:** (a) noise [6], which indicates the phenomena that the data is slightly perturbed around the subspaces (what we show is a perturbed data matrix whose columns are samples drawn from the subspaces); (b) random corruptions [7], which indicate that a fraction of random entries are grossly corrupted; (c) sample-specific corruptions (and outliers), which indicate the phenomena that a fraction of the data samples (i.e., columns of the data matrix) are far away from the subspaces.

1. noise 数据在子空间周围轻微摄动
2. 像素层面的corruptions，数据的摄动比较严重，比如加高斯噪声
3. 样本层面的corruptions，相当于outliers。a fraction of the data samples are far away from the subspaces

Liu G, Lin Z, Yan S, et al. Robust recovery of subspace structures by low-rank representation.[J]. Pattern Analysis & Machine Intelligence IEEE Transactions on, 2013, 35(1):171-84.